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IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

In re the Application of : **Docket No. 0074-26485**

POLETTI : **Examiner COREY P. CHAU**

Application No. 09/673,808 : **Art Unit 2644**

Filed: 01/12/2001 : **Confirmation No. 5524**

For: An In-Line Early Reflection Enhancement System for Enhancing Acoustics

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June 28, 2006
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Frances L. Walton
Frances L. Walton

Commissioner for Patents
Alexandria, Virginia 22313-1450

Dear Sir:

We are enclosing the following:

1) Supplemental Reply to Non-Final Action;

2) Postcard receipt.

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In the event a fee is required and is not enclosed, or the check is improper, or the fee calculation is in error, the Commissioner is authorized to charge any underpayment or credit any overpayment to the account of the undersigned attorneys, Account No. 04-1406. A duplicate copy of this sheet is enclosed.

DANN, DORFMAN, HERRELL AND SKILLMAN
A Professional Corporation

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Application No. 09/673,808
Docket No. 0074-26485

Examiner COREY P. CHAU
Art Unit 2644

SUPPLEMENTAL REPLY TO NON-FINAL ACTION

The Applicant hereby supplements the reply to the Official Action filed May 5, 2006 with the following additional remarks.

The following remarks follow the first paragraph on page 15 of the reply.

As a further explanation of the advantages provided by the Applicant's claimed invention, the Examiner is referred to the article entitled "The stability of multichannel sound systems with frequency shifting," J. Acoustical Society of America, vol. 116, No. 2, pp 853 - 871, 2004. A copy is enclosed.

The enclosed document describes Monte Carlo simulations of the stability of N-channel systems with unitary and nonunitary reverberators (same results for early reflections) without frequency shifting (Figs. 16 and 17 and Tables I and II). The loop gain per channel for a 50% risk of instability for N=8 channels is -15.2 dB for a nonunitary system (with behaviour like that of a room), and is -13.3 dB for a unitary system. In other words, the Applicant's claimed system provides 2dB more stability margin. 2dB more loop gain is significant in a sound system.

For a lower number of channels the difference is greater. For N=2, the loop gain for 50% risk is -10.2 dB for the unitary case and -13.6 dB for the non-unitary case. That is, the nonunitary system must operate with 3.4 dB less loop gain to remain stable.

These simulations clearly indicate the advantages of the Applicant's claimed system.

CONCLUSION

The Examiner is respectfully requested to consider the foregoing additional remarks and the enclosed document when reconsidering the rejection of the claims presented in this application.

Respectfully submitted,

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Enclosure

The stability of multichannel sound systems with frequency shifting

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Time-varying components are used in some multichannel sound systems designed for the enhancement of room acoustics. Time-variation can usefully reduce the risk of producing ringing tones and improve stability margins, provided that any modulation artefacts are inaudible. Frequency-shifting is one form of time-variation which provides the best case improvement in loop gain, and for which the single channel stability limit has been derived. This paper determines the stability limit for multiple channel systems with frequency-shifting by generalizing the previous single-channel analysis. It is shown that the improvement in stability due to frequency-shifting reduces with the number of channels. Simulations are presented to verify the theory. The stability limits are also compared with those of time-invariant systems, and preliminary subjective assessments are carried out to indicate useable loop gains with frequency-shifting. © *2004 Acoustical Society of America.* [DOI: 10.1121/1.1763972]

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I. INTRODUCTION

Sound systems are limited in the maximum amplification they can provide by regenerative feedback of sound from the loudspeakers to the microphones, which causes instability at high loop gains. For single channel systems with flat transfer functions in their electronic processors, the mean loop gain μ must be kept below -12 dB to avoid ringing tones or instability.[1,2] For systems with N broadband independent channels the total mean loop power gain $\mu^2 N$ rises with the number of channels, allowing a greater enhancement of early energy and reverberation time (RT).[3]

A common application of multichannel sound systems is in room acoustic enhancement systems.[1,4–7] For systems without electronic reverberation devices, the room reverberation time gain is approximately equal to the steady state sound intensity gain Γ,[1]

$$\Gamma = \frac{1}{1-\mu^2 N}. \tag{1}$$

Multichannel acoustic enhancement systems aim to provide sound quality which is indistinguishable from a natural room with enhanced acoustic properties. Therefore, they must provide a more subtle sound enhancement than that produced by typical sound systems, and with a lower risk of unnatural artefacts. For example, the tolerance of ringing tones from a sound system is lower in a classical music concert than it would be at a rock concert. For this reason, eliminating ringing tones, and other more subtle colouration effects that might occur at lower loop gains, is of primary importance. The use of multiple channels allows increased power gains with a reduced risk of colouration artefacts. Nevertheless, unnatural artefacts can still occur at high loop gains.

Aside from multiple channels, there are three other methods available for achieving increased loop gains and reducing unnatural artefacts. The first is the use of equalization.[8–13] Broadband equalization allows the loop gain to be maximized across all frequencies. Narrowband equalization allows individual ringing tones to be eliminated, but has the risk that the ringing frequency can change with small changes in microphone positions or room characteristics. A more stable approach is the use of adaptive filters which lock onto and null ringing tones.[14–16] These systems must distinguish between musical tones and ringing tones, but several commercial devices are available which suggest that this is feasible. A more comprehensive adaptive strategy is to have a filter adapt to the inverse of the loop transfer function, which renders the loop transfer function a simple delay. Alternatively, echo cancellation methods can in principle allow the cancellation of the loudspeaker to microphone signal, eliminating feedback.[17] These adaptive filter strategies rely on the robustness of an adaptive algorithm to allow the sound system to operate well above the naturally occurring stability limit.

For multichannel systems, both inverse and echo-cancellation methods require the measurement of the loop transfer function matrix which is problematic due to the large amount of memory and processing required. In addition the inverse equalization method requires calculation of the inverse of the matrix, which imposes a further large processing burden.

The second method for increasing loop gains is to employ artificial reverberation devices to provide greater reverberation gain for a given loop gain.[7,18–21] Most of these applications are in in-line systems, where the microphones are close to the sound sources. In Ref. 7 it is shown that a reverberator in a non-in-line system is the electroacoustic equivalent of a coupled room and that high reverberation gains are possible at moderate loop gains, provided that the loop gain is high enough to prevent double-sloping effects. It has also been shown that sound system stability may be improved by using unitary reverberators which produce a constant power gain with frequency, further reducing colouration effects.[22]

The third approach to controlling colouration is the use

of time-varying systems, which slowly change their transfer functions with time to prevent the build-up of ringing tones.[23–29] These systems are not as sensitive to changes in the room acoustics as narrowband equalization and are inherently more robust, and simpler, than adaptive systems.

There are several methods of producing time variation, such as delay modulation, phase- and frequency-modulation and frequency-shifting. Delay modulation provides a frequency shift which rises linearly with frequency. Therefore, it produces a reduced risk of coloration at high frequencies, but may not improve performance at low frequencies.[30] In some cases time-varying delays are incorporated into a digital reverberation device which produces a more complicated phase and amplitude variation with frequency.

Phase-modulation alters the signal phase at each frequency without affecting the amplitude. It can be carried out using time-varying allpass filters, but these produce a phase deviation that varies with frequency. A more precise method of phase modulation is achieved by taking the analytic signal and modulating it with a complex modulation of the form $\exp[j\phi(t)]$. This provides the same phase variation at all frequencies and a more consistent control of feedback across the signal bandwidth. If the complex phase function is analytic, then all components of the signal are shifted by positive frequency shifts.[31]

Frequency-shifting (FS) is a special case of analytic phase-modulation using a linear phase sinusoid. The input signal is made analytic by eliminating the negative frequencies and the frequency shifted output is the real part of the modulated signal. This single sideband technique shifts the positive frequencies in the signal by ω_0 and the negative frequencies by $-\omega_0$.[32]

The stability of sound systems with frequency-shifting has been investigated by Schroeder.[23] He developed methods for determining the maximum loop gain possible in a single channel system with and without frequency-shifting. Without frequency-shifting, his loop gain for a 50% risk of instability was -8 dB for a reverberation time, bandwidth product of 10 000. With frequency-shifting the loop gain could theoretically be increased to 2.5 dB, an increase of 10.5 dB. However, at this value the sound quality was found to be strongly affected by modulation artefacts. In practice, Schroeder determined that a 2 dB stability margin was required without FS, and a 6 dB margin was required with FS in order to avoid audible beating effects. The net benefit due to FS was then about 6 dB.

Nielsen and Svensson have recently considered the performance of periodic phase, frequency and delay modulation in single channel systems.[29] They review the theory of frequency-shifting and show that it is the most efficient method for controlling regenerative feedback since it completely smooths the loop gain and provides infinite carrier suppression.

While frequency-shifting provides the maximum possible loop gain increase, it may not produce subjectively acceptable performance for music applications. There are two reasons. First, the frequency shift is a significant fraction of a semitone at low frequencies. For example a 5 Hz shift is about a semitone at 82 Hz (low E on a guitar). Delay modulation avoids this issue but has poor stability control at low frequencies.[29] Secondly, the frequency shift is strictly positive each time the signal goes around the feedback loop (for positive modulation frequencies). The subjective effect at high loop gains is that the signal frequency rises with time which is subjectively unacceptable. However, other time-variation components which produce both positive and negative frequency shifts exhibit other problems.[29] Systems which have nonzero carrier suppression (a nonshifted component) provide a reduced stability improvement, and those which produce both positive and negative sidebands produce a net shift of zero after two or more times through the feedback loop, which also limits stability.

Nielsen and Svensson suggest that time-variation may be more successful in room enhancement systems where the sound enhancement is more subtle than that provided by sound reinforcement systems.[29] For example, frequency-shifting may offer useful control of ringing and improve the loop gain provided that the shift is small enough to avoid low-frequency problems and the loop gain increase is not too great.

The analysis of time-varying systems has to date concentrated on single channel systems, although some experiments on a four-channel time-varying system are reported in Ref. 30. However, no analysis of the theoretical performance gains that can be achieved using time-variation in multichannel systems has been carried out.

In this paper, an analysis of frequency-shifting in multichannel systems is undertaken. Frequency-shifting is the simplest form of time-variation to analyze and, as discussed above, it provides the best-case increase in loop gain. The analysis of the multichannel case follows that in Schroeder's paper and Ref. 29. The multichannel extension of this work is based on a norm approximation for the total power gain of the loop transfer function matrix.

Two forms of simulations are carried out to verify the theory. Monte Carlo stability simulations are used first to verify the theoretical loop gain limits, and then a time-varying system simulator based on digital reverberators is used to allow a more detailed investigation of the behavior of sound systems with frequency-shifting.

Finally, the frequency-shifting stability limits are compared with the stability limits for time-invariant systems and subjective tests are carried out in order to estimate the practically useable loop gain for both cases. The practical increase in loop gain brought about by frequency-shifting is then estimated.

II. THEORETICAL STABILITY LIMITS

A. Review of power gains

The analysis of multichannel frequency-shifting is based on the power gain of the loop transfer function matrix. The power gain is therefore briefly reviewed here.

Consider a room containing N loudspeakers and microphones. Assume that the distances between loudspeakers and microphones are large enough so that the reverberant sound transmission dominates the direct sound. Each transfer function $H_{nm}(\omega)$ then has real and imaginary parts which are

normally distributed (with transducer positions) with zero mean and equal variances $\sigma_I^2=\sigma_R^2$.[23] The magnitude $|H_{nm}(\omega)|$ is Rayleigh distributed.

If a signal with power spectral density (psd) $S_{uu}(\omega)$ is input to $H_{nm}(\omega)$, the output psd is[33]

$$S_{vv}(\omega)=|H_{nm}(\omega)|^2 S_{uu}(\omega). \tag{2}$$

The power gain at each frequency is then $|H_{nm}(\omega)|^2$.

The mean power gain may be found as the expected value of $|H_{nm}(\omega)|^2$ over the ensemble of all possible microphone and loudspeaker positions. $|H_{nm}(\omega)|^2=H_{nmR}^2(\omega)+H_{nmI}^2(\omega)$ is χ-squared distributed with two degrees of freedom and so[34]

$$E\{|H_{nm}(\omega)|^2\}=2\sigma_R^2. \tag{3}$$

Assume further without loss of generality that the mean power gain of each transfer function is unity ($\sigma_R^2=1/2$), and that the output of each microphone preamplifier is attenuated by a factor μ to control stability.

Consider now the matrix of all transfer functions, $\mathbf{H}$. Assume that N uncorrelated signals with the same power spectrum $S_{uu}(\omega)$ are applied to the N inputs of $\mathbf{H}$. (This is realistic if the microphones receive predominantly reverberant energy and have the same sensitivities and preamplifier gains.) The power at each output of $\mathbf{H}$ is then the sum of the powers arriving from each input

$$S_{vvn}(\omega)=\sum_{m=1}^{N}|H_{nm}|^2 S_{uu}(\omega) \tag{4}$$

and the total output power is

$$S_{vv}(\omega)=\sum_{m=1}^{N}\sum_{n=1}^{N}|H_{nm}|^2 S_{uu}(\omega)=\|\mathbf{H}\|^2 S_{uu}(\omega), \tag{5}$$

where $\| \|$ denotes the Frobenius norm of $\mathbf{H}$. The total input power is $NS_{uu}(\omega)$. Therefore the total power gain is

$$\Gamma_H=\frac{1}{N}\|\mathbf{H}\|^2. \tag{6}$$

The expected value of the power gain is, with the assumptions above

$$E\{\Gamma_H\}=\frac{1}{N}N^2=N. \tag{7}$$

This is scaled by μ^2 to control stability.

B. Frequency-shifting and analytic signal analysis

In frequency-shifting a real signal $s(t)$ is made complex by deriving an imaginary part $\hat{s}(t)$ which is the Hilbert transform of $s(t)$,[33]

$$s_a(t)=s(t)+j\hat{s}(t). \tag{8}$$

This analytic signal has no negative frequency components. Frequency-shifting is carried out by multiplying the analytic signal by $\exp(j\omega_0 t)$ and taking the real part of the result

$$\mathrm{Re}\{s_a(t)e^{j\omega_0 t}\}=s(t)\cos(\omega_0 t)-\hat{s}(t)\sin(\omega_0 t), \tag{9}$$

which reintroduces negative frequency components such that the spectrum is Hermitian.[33]

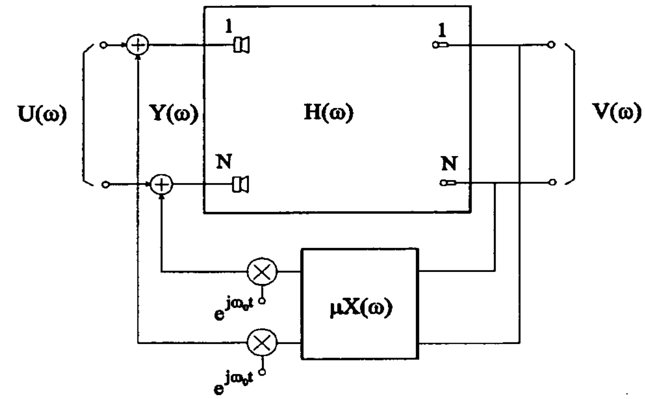

FIG. 1. Multichannel sound system with frequency-shifting, assuming an analytic input signal.

The analytic signal is often used in the analysis of time-invariant physical systems, because any real, physical signal is simply the real part of the associated analytic signal. The same approach is followed here. However, in the case of time-varying systems, the analytic signal is modulated and this modulation may produce negative frequency components. In frequency-shifting, this occurs for large, negative frequency shifts. In the application of interest here the frequency shift is small, and the room transfer functions have magnitudes that reduce to zero at zero Hz (since the loudspeaker responses are zero at dc). Therefore the modulated analytic signal will also remain analytic for negative frequency shifts. Hence, the analysis of stability with frequency-shifting may be carried out using analytic signals, and the spectrum of the output signal for the associated real signal is simply the Hermitian part of the analytic signal output.

C. Multichannel systems with frequency-shifting

Consider the regenerative system shown in Fig. 1. The transfer function matrix from the N loudspeaker inputs to the N microphone outputs is $\mathbf{H}(\omega)$. For simplicity, we have omitted the use of separate input and output transfer functions as in Ref. 2, since this has no effect on the stability. The input signals are applied directly to the N system loudspeakers. The stability of the system will be examined by considering the vector of spectra at the input to the room, Y. In the time invariant case, this is given by

$$Y=[\mathbf{I}-\mu\mathbf{X}\mathbf{H}]^{-1}U=\mathbf{H}^{-1}[\mathbf{I}-\mu\mathbf{H}\mathbf{X}]^{-1}\mathbf{H}U. \tag{10}$$

The stability of the time-invariant system is governed by the scaled eigenvalues of $\mathbf{XH}$, which are the same as those of $\mathbf{HX}$.[35] The output vector is $V=\mathbf{H}Y$ which has the same stability criteria as Y.

For the time-variant case, we assume for simplicity of analysis that all microphones signals are frequency shifted by the same modulation function $\exp[j\omega_0 t]$. As discussed above, we assume an input signal $U(\omega)$ which is analytic. The vector of spectra $Y(\omega)$ at the inputs to the room with frequency-shifting may then be written

$$Y(\omega)=U(\omega)+\mu\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)Y(\omega-\omega_0) \tag{11}$$

Substituting for $Y(\omega-\omega_0)$ from the same equation

$$\begin{aligned}Y(\omega)&=U(\omega)+\mu\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)[U(\omega-\omega_0)\\&\quad+\mu\mathbf{X}(\omega-2\omega_0)\mathbf{H}(\omega-2\omega_0)Y(\omega-2\omega_0)]\\&=U(\omega)+\mu\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)U(\omega-\omega_0)\\&\quad+\mu^2\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)\mathbf{X}(\omega-2\omega_0)\\&\quad\times\mathbf{H}(\omega-2\omega_0)Y(\omega-2\omega_0).\end{aligned}\tag{12}$$

This procedure may be continued, and the vector of spectra is given by the infinite series

$$\begin{aligned}Y(\omega)&=U(\omega)+\mu\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)U(\omega-\omega_0)\\&\quad+\mu^2\mathbf{X}(\omega-\omega_0)\mathbf{H}(\omega-\omega_0)\mathbf{X}(\omega-2\omega_0)\mathbf{H}(\omega-2\omega_0)\\&\quad\times U(\omega-2\omega_0)\cdots+\mu^M\left[\prod_{m=1}^{M}\mathbf{X}(\omega-m\omega_0)\right.\\&\quad\left.\times\mathbf{H}(\omega-m\omega_0)\right]U(\omega-M\omega_0)+\cdots.\end{aligned}\tag{13}$$

If the system is to be stable the output power must be finite for a finite input power. The total power in all channels that contributes to Y after M iterations through the feedback loop is the total input power multiplied by the power gain

$$\Gamma_M=\frac{\mu^{2M}}{N}\left\|\prod_{m=1}^{M}\mathbf{X}(\omega-m\omega_0)\mathbf{H}(\omega-m\omega_0)\right\|^2.\tag{14}$$

D. Stability analysis for unitary systems

We consider first the case where there is no second transfer function matrix $\mathbf{X}$, and each output is connected directly via a frequency-shifter and loop gain to the corresponding input. The power gain becomes

$$\Gamma_M=\frac{\mu^{2M}}{N}\left\|\prod_{m=1}^{M}\mathbf{H}(\omega-m\omega_0)\right\|^2.\tag{15}$$

This expression is more problematic than the single channel case, because the squared norm of the product of matrices is not equal to the product of the squared norms unless $N=1$. It is known that for products of matrices, the following inequality holds:[35]

$$\left\|\prod_{m=1}^{M}\mathbf{H}(\omega-m\omega_0)\right\|^2<\prod_{m=1}^{M}\|\mathbf{H}(\omega-m\omega_0)\|^2,\tag{16}$$

but this does not provide a tight enough bound to allow the stability to be accurately determined. However, for the case where the frequency shifted matrices are uncorrelated, consisting of entries which are complex, zero-mean, and whose real and imaginary parts are normally distributed, it can be shown (Appendix A) that

$$\left\|\prod_{m=1}^{M}\mathbf{H}(\omega-m\omega_0)\right\|^2\approx\frac{1}{N^{[M-1]}}\prod_{m=1}^{M}\|\mathbf{H}(\omega-m\omega_0)\|^2.\tag{17}$$

This approximation is good for large N, but is less accurate for small N, becoming least accurate for $N=2$.

The transfer functions in $\mathbf{H}$ have identical real part and squared-magnitude covariances of[36]

$$\rho(\omega_0)=\frac{1}{1+\left(\frac{\omega_0 T}{13.8}\right)^2},\tag{18}$$

where T is the reverberation time. The covariance is 0.2 for $\omega_0=27.6/T$ and the norm approximation should be reasonably accurate (up to the accuracy governed by N) for frequency shifts greater than this value. For a reverberation time of 1 s this requires a frequency shift of 4.4 Hz.

Using the norm approximation, the power gain

$$\Gamma_M\approx\frac{\mu^{2M}}{N^M}\prod_{m=1}^{M}\|\mathbf{H}(\omega-m\omega_0)\|^2\tag{19}$$

must reduce to zero as M tends to infinity. Writing the approximation in dB

$$\begin{aligned}10\log(\Gamma_M)&=M\mu_{\mathrm{dB}}-M10\log(N)\\&\quad+\sum_{m=1}^{M}10\log[\|\mathbf{H}(\omega-m\omega_0)\|^2]\to-\infty\end{aligned}\tag{20}$$

as M tends to infinity. For large M, the summation tends to the scaled mean of the norm in dB,

$$\begin{aligned}10\log(\Gamma_M)&=M[\mu_{\mathrm{dB}}-10\log(N)+\overline{10\log[\|\mathbf{H}(\omega)\|^2]}]\\&\to-\infty,M\to\infty\end{aligned}\tag{21}$$

the stability requirement then becomes

$$\mu_{\mathrm{dB}}<10\log(N)-\overline{10\log[\|\mathbf{H}(\omega)\|^2]}.\tag{22}$$

In Appendices B and C it is shown that

$$\begin{aligned}\overline{10\log[\|\mathbf{H}(\omega)\|^2]}&=\frac{10}{\ln(10)}\Psi(N^2)\\&=\frac{10}{\ln(10)}\left[\sum_{k=1}^{N^2-1}\frac{1}{k}-C\right],\end{aligned}\tag{23}$$

where Ψ (.) is the psi function,[37] and so the stability limit is

$$\mu_{\mathrm{dB}}<10\log(N)-\frac{10}{\ln(10)}\Psi(N^2).\tag{24}$$

This is the stability requirement for an N channel system with identical frequency-shifting in each channel, assuming that the norm approximation is accurate. For $N=1$ it yields $\mu_{\mathrm{dB}}<2.5$ as in the original analysis.[23] The maximum loop gain with frequency-shifting reduces with N. For example, it is -12 dB for a 16 channel system. This is in keeping with the maximum loop gain per channel without frequency-shifting, which also decreases with N.

The stability limit also applies to multichannel systems with unitary processors in their feedback loops, since a unitary matrix does not alter the loop power gain. Specifically, if $\mathbf{X}$ is a unitary matrix, then

$$\|\mathbf{XH}\|^2=\mathrm{Tr}\{(\mathbf{XH})^\dagger(\mathbf{XH})\}=\mathrm{Tr}\{\mathbf{H}^\dagger\mathbf{X}^\dagger\mathbf{XH}\}=\|\mathbf{H}\|^2,\tag{25}$$

where $\dagger$ denotes the conjugate transpose and Tr the trace function.

E. Stability analysis with finite M

The analysis in the previous section requires that the sound regenerate through the feedback loop a large number of times. This requirement is satisfied in wideband systems with small frequency shifts. However for bandlimited systems, and high-frequency input signals, the number of iterations M will be smaller than for low frequency signals. It is therefore instructive to consider the stability analysis for a finite M. The analysis will also be useful for verification of the stability equation using Monte Carlo simulations.

Consider Eq. (20). The stability condition is obtained by assuming that the summation tends to the average of the norm in dB, scaled by M. An alternative viewpoint is that the summation is a random variable (with microphone and loudspeaker position) which tends to a normal variable as M increases. Provided that ω_0 is large enough each term in the sum is independent of the others, and has the form $10\log(y_m)$, where $y_m=\|H(\omega-m\omega_0)\|^2$. The squared norm y_m is χ-squared distributed with $n=2N^2$ degrees of freedom. The mean of the squared norm in dB is given by Eq. (23) and the variance is (Appendix B)

$$\sigma_y^2=\mathrm{var}\{10\log(y)\}=\left(\frac{10}{\ln(10)}\right)^2\zeta(2,N^2)$$
$$=\left(\frac{10}{\ln(10)}\right)^2\left[K-\sum_{k=1}^{N^2-1}\frac{1}{k^2}\right], \qquad (26)$$

where $K=1.644924$ and $\zeta(2,N^2)$ is the Riemann zeta function (Appendix C).

The sum of M terms, by the Central limit theorem, tends to a normal density with mean $M\bar{y}$ and variance $M\sigma_y^2$. If y is scaled by a loop gain μ^2 the probability of instability is then

$$\Pr\{\mathrm{inst},\mathbf{H}\}(\mu_{\mathrm{dB}},N,M)=\Pr\left\{\sum_{m=0}^{M-1}10\log(\mu^2 y_m)\right.$$
$$\left.>10M\log(N)\right\} \qquad (27)$$

[this is equivalent to saying that the power gain of the Mth term in Eq. (13) is greater than one]. Hence

$$\Pr\{\mathrm{inst},\mathbf{H}\}(\mu_{\mathrm{dB}},N,M)$$
$$=\frac{1}{2}-\frac{1}{2}\mathrm{erf}\left(\sqrt{\frac{M}{2}}\frac{10\log(N)-\mu_{\mathrm{dB}}-\frac{10}{\ln(10)}\Psi(N^2)}{\frac{10}{\ln(10)}\sqrt{\zeta(2,N^2)}}\right), \qquad (28)$$

where erf is the error function. The 50% risk of instability occurs when the argument of the erf function is zero, which yields the stability limit in Eq. (24). As M increases, the transition from 0 to 1 through the 50% value as μ increases becomes increasingly rapid, and for large M the stability limit is sufficient to characterize the stability of the system.

F. Stability analysis for systems including reverberators

We now consider the case where the second transfer function matrix $\mathbf{X}$ represents a nonunitary reverberator. Many sound systems include reverberators to enhance the sound transmitted into the room. This increases the variance of the power gain of the feedback loop. In time-invariant systems this means that the loop gain must be reduced to maintain stability. With frequency-shifting, it has been shown that for single channel systems this increased variance allows a higher loop gain to be achieved than that without a reverberator present.[29]

We will assume that the reverberator has the same statistical norm properties as the room matrix $\mathbf{H}$ but is scaled by $1/\sqrt{N}$ to have unit power gain, which allows comparison with the unitary case. Equivalently, $\mathbf{X}$ is statistically identical to $\mathbf{H}$ and the loop gain is scaled by $1/\sqrt{N}$.

The stability requirement is given by Eq. (22), but the matrix $\mathbf{H}$ is replaced by $\mathbf{XH}$, and the division by $\sqrt{N}$ introduces an additional $10\log(N)$ to the loop gain

$$[\mu_{\mathrm{dB}}-10\log(N)]<10\log(N)-\overline{10\log[\|\mathbf{X}(\omega)\mathbf{H}(\omega)\|^2]}. \qquad (29)$$

Treating $\mathbf{H}$ and $\mathbf{X}$ as independent, identically distributed matrix random variables, and using the norm approximation

$$\overline{10\log[\|\mathbf{XH}\|^2]}\approx\overline{10\log\left[\frac{1}{N}\|\mathbf{X}\|^2\|\mathbf{H}\|^2\right]}$$
$$=20\,\overline{\log[\|\mathbf{H}\|^2]}-10\log(N) \qquad (30)$$

the stability requirement is

$$\mu_{\mathrm{dB}}+20\,\overline{\log[\|\mathbf{H}(\omega)\|^2]}<30\log(N). \qquad (31)$$

Substituting for $20\,\overline{\log[\|\mathbf{H}(\omega)\|^2]}$ from Appendix B yields the stability criterion

$$\mu_{\mathrm{dB}}<30\log(N)-\frac{20}{\ln(10)}\Psi(N^2). \qquad (32)$$

The analysis for a finite number of iterations through the feedback loop may be carried out in a similar manner to the previous case. Applying the norm equality to $\mathbf{XH}$ it may be shown that the mean with normalized $\mathbf{X}$ is

$$\overline{10\log\left\|\frac{1}{\sqrt{N}}\mathbf{XH}\right\|^2}=-20\log(N)+\frac{20}{\ln(10)}\Psi(N^2). \qquad (33)$$

The variance of the norm in dB is unaffected by the $1/\sqrt{N}$ scaling, and is twice that without a reverberator [see Eqs. (30) and (26)]

$$\mathrm{var}\left\{10\log\left\|\frac{1}{\sqrt{N}}\mathbf{XH}\right\|^2\right\}=2\left(\frac{10}{\ln(10)}\right)^2\zeta(2,N^2). \qquad (34)$$

The stability for M iterations is then

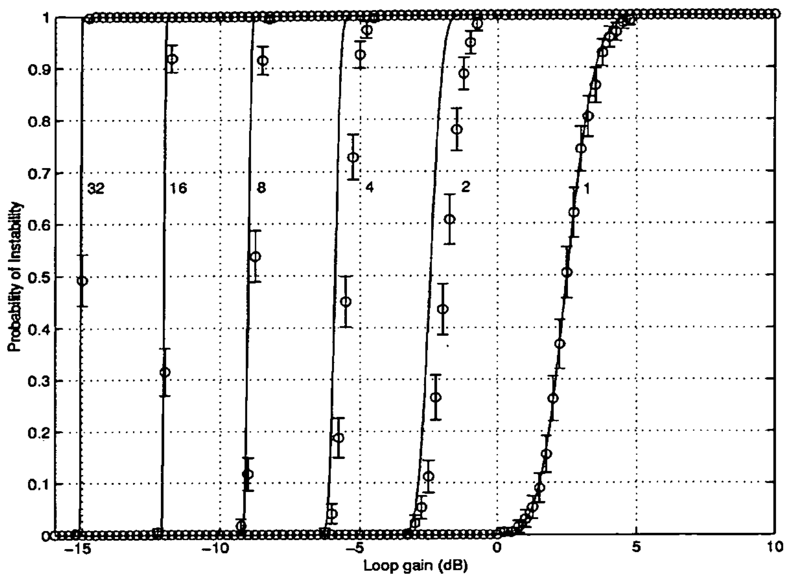

FIG. 2. Monte Carlo Simulation results for sound system with unitary feedback for $N=1-32$.

$$\Pr\{\text{inst},\mathbf{XH}\}(\mu_{\text{dB}},N,M)$$

$$=\frac{1}{2}-\frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{M}}{2}\frac{30\log(N)-\mu_{\text{dB}}-\frac{20}{\ln(10)}\Psi(N^2)}{\frac{10}{\ln(10)}\sqrt{\zeta(2,N^2)}}\right). \tag{35}$$

The 50% risk loop gain is then given by the right-hand side of Eq. (32).

III. STABILITY SIMULATIONS

A. Introduction

The validity of the theoretical stability limits were investigated using two forms of simulation. In the first, Monte Carlo simulations were carried out to assess Eqs. (28) and (35) for a finite value of M. The 50% risk values of these simulations should match the stability limits in Eqs. (24) and (32). In the second approach a digital reverberator was used to represent a room and feedback was applied from N outputs to N inputs via frequency shifters. A second reverberator could be optionally included in the feedback loop to simulate the inclusion of a reverberator in a sound system.

B. Monte Carlo simulations

The transfer function matrix between a loudspeaker and microphone in a room has real and imaginary parts which are normally distributed, provided that the direct sound level is small compared to the reverberant sound level. The correlation functions of the real and imaginary parts are as given in Eq. (18). Therefore, complex random processes may be generated using an autoregressive (AR) process which produce the same statistical and correlation properties as the transfer functions in rooms.[2,38]

If an N channel sound system is used, and the microphones and loudspeakers are further away from each other than the spatial correlation distance,[39] then independent autoregressive processes with the same statistics and correlation properties may be used to model the transfer function matrix.

Hence, to determine the stability of multichannel sound systems with frequency-shifting, an AR process is used to produce a set of transfer function matrices at equally spaced frequencies. The set of transfer function matrices are scaled by the loop gain. The values of the scaled transfer function matrices at M frequencies spaced at f_0, where f_0 is the frequency shift, are multiplied together. The squared norm of the product is then found and the power gain calculated. If the power gain exceeds one, the system is assumed to be unstable. The simulation is then repeated with a new AR process. The probability of instability at the given loop gain may be determined from K such simulations, and the variance calculated.[2] For systems including reverberators, a second AR process is used to simulate the reverberator.

Monte Carlo simulations are shown in Fig. 2 for a room with a reverberation time of 1 second, for numbers of channels in powers of 2 from 1 to 32 and for $M=50$. The number of trials was 400. The frequency shift was 10 Hz, for which the correlation between samples is 0.05. The predicted stability risk is closely matched by theory for $N=1$, where the norm approximation is exact. For $N=2$ the theoretical 50% value is 0.5 dB too low. For $N=4$ and 8 the error is 0.4 and 0.25 dB, respectively. For $N=16$ and 32 the error is negligible. Hence, the norm approximation produces a worst-case error of half a dB for $N=2$ and the error reduces with N.

For smaller frequency shifts, the assumption of independent variables in the summation of Eq. (20) becomes less correct. The slope of the resulting stability curve reduces slightly, diverging from the theoretical curve predicted by Eq. (28) for small and large probabilities, and particularly for small N where the slope is lower. However, the deviation is small for probabilities near 50% and the 50% stability limit

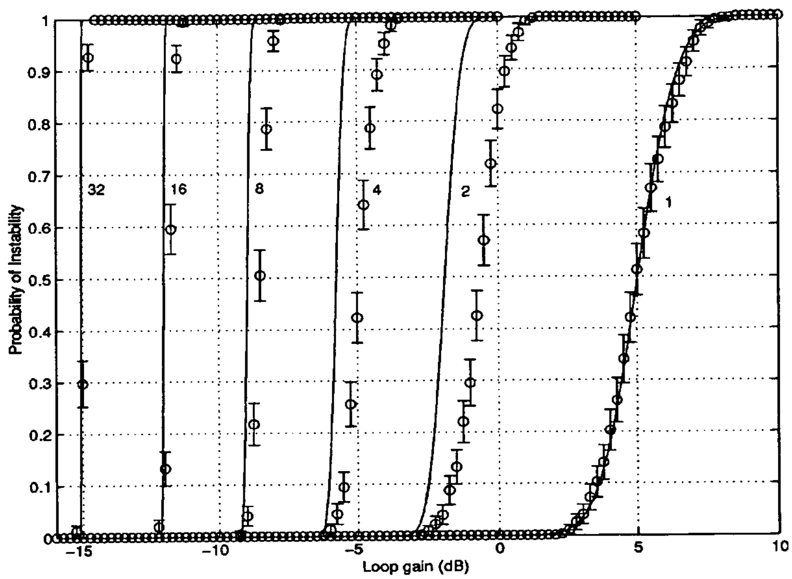

FIG. 3. Monte Carlo Simulation results for sound system with a reverberator in the feedback path for $N = 1-32$.

remains close to the theoretical stability limit predicted by Eq. (28).

The equivalent simulations for a sound system including a multichannel reverberator with natural statistics are shown in Fig. 3. The $N=1$ results are again accurate. The errors for $N=2$, 4, 8, and 16 are 1.3, 0.9, 0.5, and 0.25 dB, respectively. These are larger than the results without a reverberator, because the norm approximation was applied twice to produce the probability of instability.

C. Reverberator-based simulations

In order to further investigate the accuracy of the stability limits derived above, a second simulator was developed. The transfer function matrix of a room was simulated as a time-invariant, 24-channel reverberator using 24 delay lines cross-coupled via an orthonormal matrix.[7] This provided a fully cross-coupled 24×24 matrix of transfer functions with Rayleigh magnitude statistics. Feedback was then applied around $N<24$ channels of the reverberator and frequency-shifting was implemented in the feedback loop. A second reverberator was implemented in the feedback loops as required. This second reverberator could either simulate the Rayleigh statistics of rooms, or could be of a unitary design. The majority of simulations were carried out with no secondary reverberator, or with a standard nonunitary reverberator. Some unitary reverberator simulations were later run to check that the stability limits were the same as the no-reverberator case.

A block diagram of the simulator is shown in Fig. 4. As discussed, this simulator uses an analytic input signal, and so the analytic filter $A(\omega)$ is placed at the input to eliminate the negative frequencies in any applied real signal. The signals in the reverberator are thus complex. The envelope of a given output is obtained from its low-pass filtered magnitude, and the real part of two adjacent channels written to a stereo wave file for subjective assessments, discussed in Sec. IV.

The continuous-time analytic filter $A(\omega)$ has the theoretical complex impulse response

$$h_a(t)=\delta(t)+j\frac{1}{\pi t}. \tag{36}$$

In practice, a discrete, finite response, bandlimited form of the analytic filter is required. The finite response means that

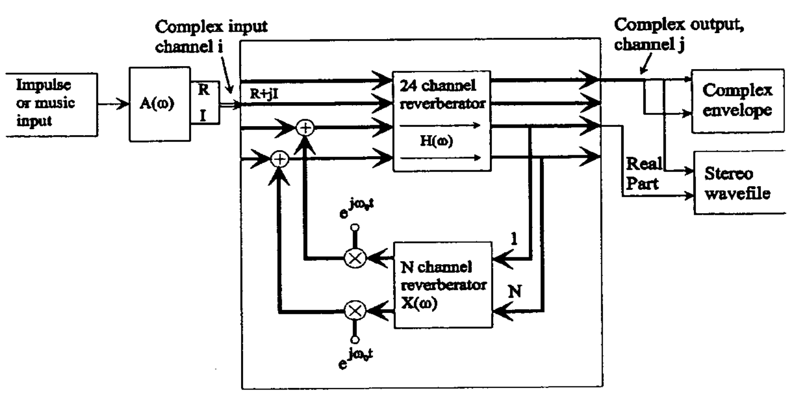

FIG. 4. Reverberator-based frequency shifting simulator. $X(\omega)$ is bypassed for simulations without a secondary reverberator.

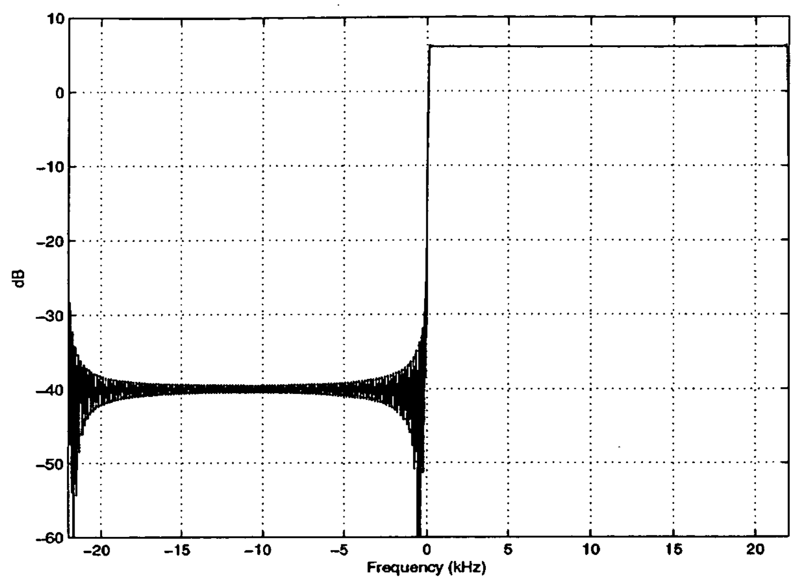

FIG. 5. Magnitude of the analytic filter response.

the analytic filter has a low frequency cutoff below which the magnitude of the response reduces. A 512 tap finite impulse response filter was designed by specifying a desired complex filter response with a gain of two for positive frequencies and a linear transition from a transition frequency of 100 Hz to an attenuation of -40 dB at zero frequency. The filter taps were then obtained from a least squares fit to the desired response. The resulting magnitude response is shown in Fig. 5.

The "naturalness" of the reverberators may be verified by showing that they produce ideal statistics in the frequency domain, and that they have a high echo density in time with an absence of flutter effects. The squared norm of the reverberator transfer function matrices were close to the theoretical probability densities for all N. For example, the measured probability density of the squared norm of the transfer function matrix for 8 channels is shown in Fig. 6 together with the theoretical χ-squared density. The measured density is closely similar to the theoretical.

The echo density was evaluated by listening to the impulse response. The response sounded smooth without any noticeable flutter effects. The envelope of the decay was exponential and matched the required RT. No damping was applied to the reverberator, so that its reverberation time was constant with frequency. A reverberation time of 1.0 s was used in both the main and the secondary reverberator. The theoretical stability limit is not affected by the RT, and the RT of the reverberators was controlled by altering their internal delay times which did not affect their norm statistics. A frequency shift of 5 Hz was used, producing a correlation between adjacent samples of 0.16. The sample rate was 44.1 kHz, and first-order 20 Hz high pass and 20 kHz low pass

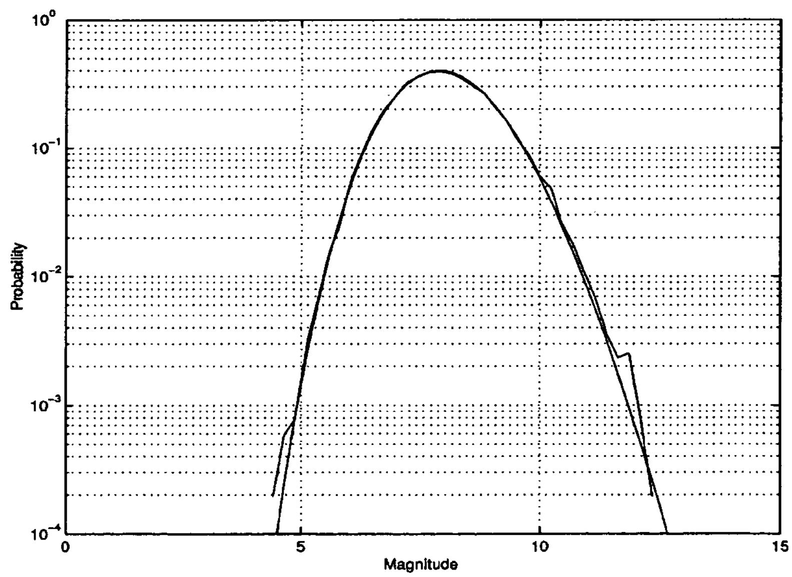

FIG. 6. Theoretical and measured probability density of the power gain of an 8 channel reverberator transfer function matrix.

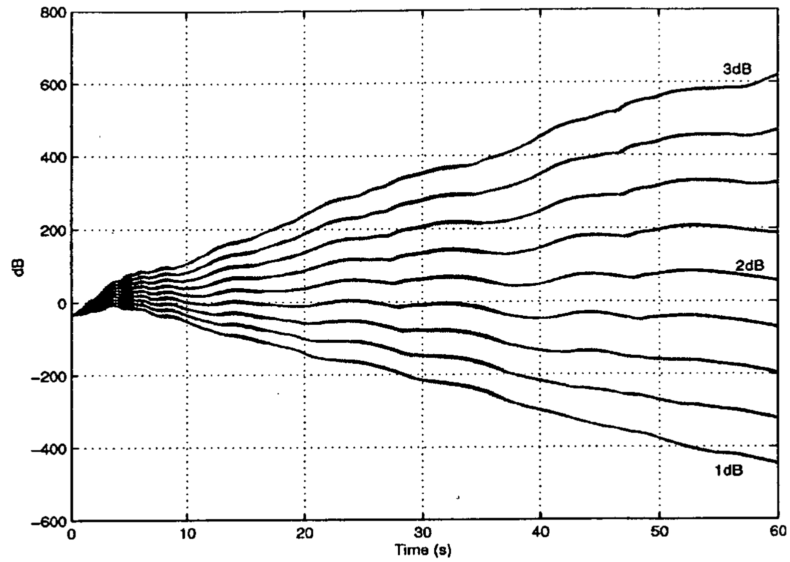

FIG. 7. Set of output envelopes for a single channel system without reverberator and with loop gains from 1 dB to 3 dB in 0.25 dB steps.

filters were implemented in each channel to prevent the appearance of negative frequency components.

Simulations were run for $N=1$, and for even channels from 2 to 16, and for a number of loop gains from below to above the relevant theoretical stability limits in Eqs. (24) and (32). A duration of 60 s was used to allow the envelope to be observed for a considerable time. For high numbers of channels the decays were linear and 30 s were sufficient to establish stability. In all cases the input was a unit delta function which, upon being filtered by the analytic filter, produces the analytic filter impulse response.

For low channel numbers the loop gain was incremented in 0.25 dB steps, but for some of the higher channel numbers more closely spaced increments of 0.1 and 0.05 dB were possible. For each simulation the squared magnitude of the analytic output signal was found. This was low-pass filtered and then decimated by a factor of 128 to reduce the data size. The set of power envelopes for the range of loop gains was then plotted. The stability limit was estimated as the loop gain which produced the envelope with slope closest to zero.

A set of power envelopes obtained for $N=1$ are shown in Fig. 7, for loop gains from 1 to 3 dB in 0.25 dB steps. The envelope initially rises for all loop gains, but then reduces with time for loop gains below 2 dB, suggesting that 2 dB is the stability limit. This is close to the theoretical limit of 2.5 dB. Nielsen and Svensson produce a maximum loop gain of 0 dB from measurements.[29] However, they noted that large temporary level increases occurred which saturated their equipment, particularly for low frequency shifts. Figure 7 verifies that these large increases occur even for the relatively large shift of 5 Hz (and larger level increases were found for simulations with lower frequency shifts). This suggests that, in practice, the 2.5 dB limit is not achievable because any sound system will have insufficient dynamic range to handle the signal amplitude at such loop gains. It is also highly likely that the sound quality will be unacceptable at loop gains approaching the stability limit in any case.

A second simulation for $N=1$ is shown in Fig. 8, using lower loop gains ranging from -1 to $+1$ dB, and looking at the first 2 s of the response. The output envelope slope is zero or negative for loop gains of 0 dB and lower, and so saturation of audio systems is unlikely. Hence 0 dB is a practical stability limit, and is consistent with the results measured by Nielsen and Svensson.

The output envelopes for a 16 channel system with no secondary reverberator are shown in Fig. 9, for loop gains ranging from -12.15 to -11.85 dB in 0.05 dB steps. In this case, there is no large initial buildup of level. The envelopes are approximately linear with time, and the stability limit is in the vicinity of -11.95 dB, which is 0.08 dB from the theoretical value of -12.03 dB. The simulations in this case produce a stability limit very close to the theoretical prediction.

The complete set of simulation-derived and theoretical stability limits without a reverberator are shown in Fig. 10. Also included are Monte Carlo results for the 50% risk loop gain. The $N=1$ simulator result is 0.5 dB lower than theory, whereas the Monte Carlo result is closer. Since both simulations include the correlations between samples of the transfer function matrix, finite correlations between adjacent products does not explain this discrepancy. It may be explained by the approximation of the sum in Eq. (20) by the exact mean. For $N=2$ both simulations are about 0.5 dB higher than theory and the error tends to reduce with N. Also included in Fig. 10 is the 50% risk of instability limit without time variation found using the methods in Ref. 2. This allows the theoretical improvement in loop gain produced by frequency-shifting to be determined. For $N=1$ the theoretical increase is about 11 dB and as N increases the improvement reduces to around 3 dB for the 16 channel case. Thus, it may be concluded that time-variation produces a reducing benefit with the number of channels. The reason for this is that for large N, the statistics of the loop power gain become increasingly ideal. From Eqs. (B3) and (B4), the mean of the squared norm of H is N^2 (for $2\sigma_R^2=1$) and the variance is

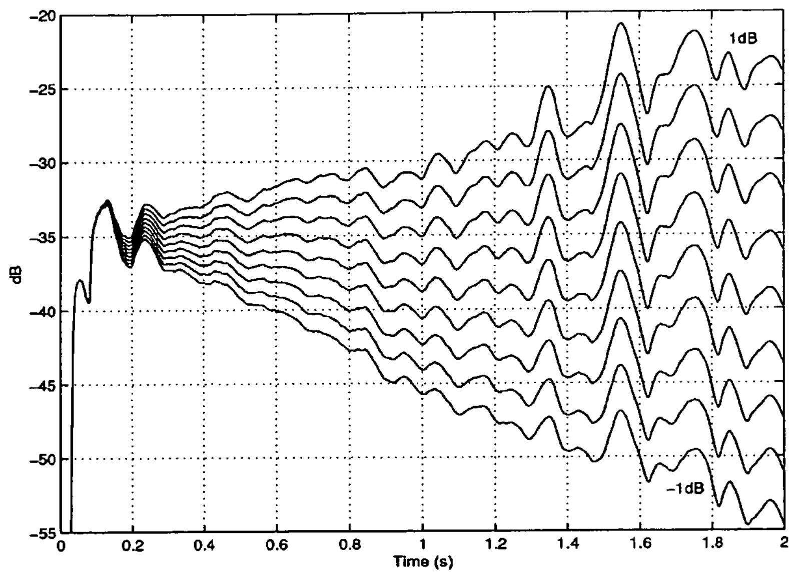

FIG. 8. Set of output envelopes for a single channel system without reverberator and with loop gains from − 1 dB to 1 dB in 0.25 dB steps.

$N^2/4$. The relative deviation of the squared norm may be quantified by the ratio of the standard deviation to the mean, which is

$$\varepsilon = \frac{1}{2N}. \tag{37}$$

The deviation of the loop power gain thus reduces with N, and there is a reducing variation for time variation to exploit.

Simulations were also carried out which included a nonunitary reverberator with the same statistics as those of a room. The output envelopes showed a reduced slope variation with loop gain compared to the no-reverberator case. An example for $N=16$ is shown in Fig. 11 for loop gains varying in 0.1 dB steps (twice the stepsize of that in Fig. 9). The envelope slopes have about half the variation with loop gain of those in Fig. 9. This is explained by Fig. 3, which shows a slower transition from stability to instability compared to Fig. 2. The stability limit is about − 11.8 dB. Because of the slower transition from instability to stability, time varying systems with nonunitary reverberators will require a larger stability margin than systems without reverberators.

The complete set of stability limits for systems with nonunitary reverberators is shown in Fig. 12. For $N=1$ the Monte Carlo result is closer to theory than the reverberator simulation, but the remaining results follow a similar trend, with around 1 dB error for $N=2$ and a reducing error with N. However, the closeness of theory and measurement is not as good as the previous case, due to the additional norm approximation required in the derivation of the theoretical stability limit.

The stability limits for the case of no time variation are also included in Fig. 12. (These values are discussed more

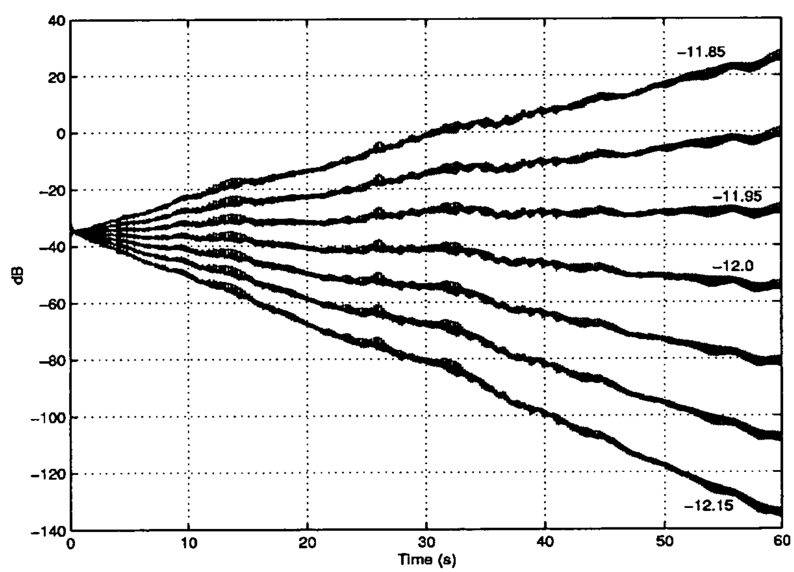

FIG. 9. Set of output envelopes for a sixteen channel system without reverberator and with loop gains from − 12.15 dB to − 11.85 dB in 0.05 dB steps.

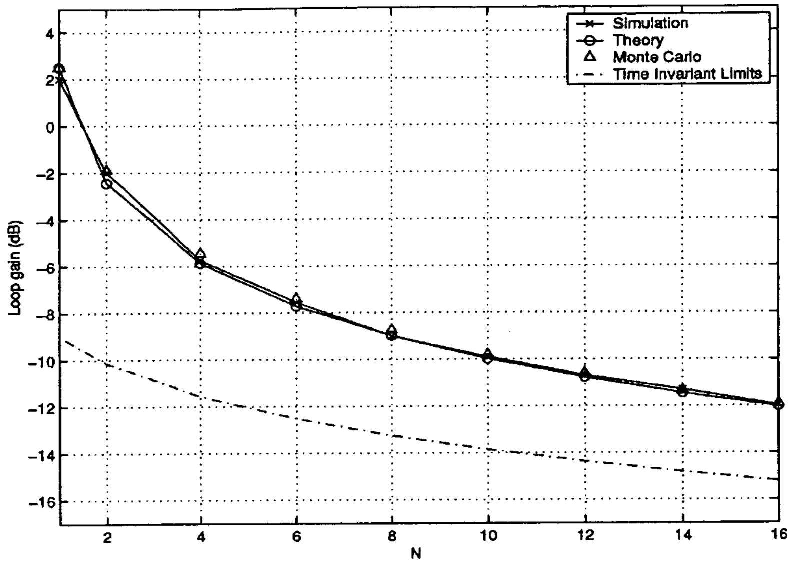

FIG. 10. Stability limits for multichannel sound systems without reverberators, including Monte Carlo results and theoretical time-invariant loop gain limits for 50% risk of instability.

fully in Sec. III D.) The results show that frequency-shifting allows the mean loop gain to be increased by 18 dB for the single channel case, which is 7 dB larger than the case with no reverberator.

For $N>1$ the maximum loop gain limits are approximately the same for systems with and without reverberators, and the two limits become closer as N increases. The increased loop gain enhancement for systems with reverberators at low channel numbers occurs because these systems have an inherently poorer performance without frequency-shifting, due to the greater variance of their loop transfer functions. Time-variance allows these systems to bring their loop gains up to the same levels as unitary systems, and to exceed them at low channel numbers.

The calculation of the theoretical loop gain limits in Sec. II requires that the same frequency shift is used in all channels. In practice, different frequency shifts could be used. To investigate this case a simulation was run for a 16 channel system with the same parameters as in Fig. 9, but with normally distributed frequency shifts varying from 3.27 Hz to 6.31 Hz, with a nominal mean of 5 Hz. The power envelopes are shown in Fig. 13, which may be compared to the envelopes in Fig. 9. Each decay curve has a slightly increased slope, and this is probably due to the fact that the frequency shift in some channels is lower than 5 Hz. However the stability limit of about -12.02 dB is only slightly higher than the -11.95 dB limit in Fig. 9. Hence the stability limit does not appear to be significantly affected by using different frequency shifts per channel. However, if too many channels have low frequency shifts, it is likely that the stability performance will degrade more significantly. Furthermore, high frequency shifts will produce more noticeable modulation artefacts.

In practical multichannel sound systems, variable rever-

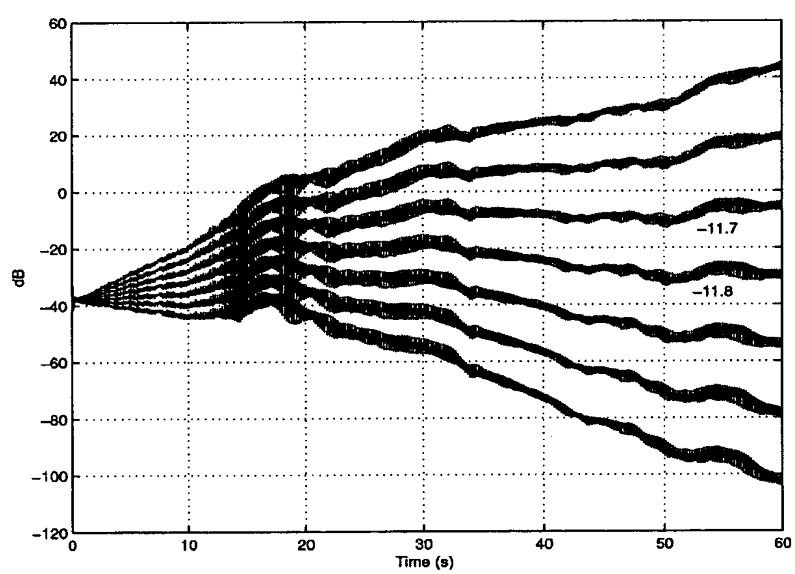

FIG. 11. Set of output envelopes for a sixteen channel system with reverberator and with loop gains from -12.1 dB to -11.5 dB in 0.1 dB steps.

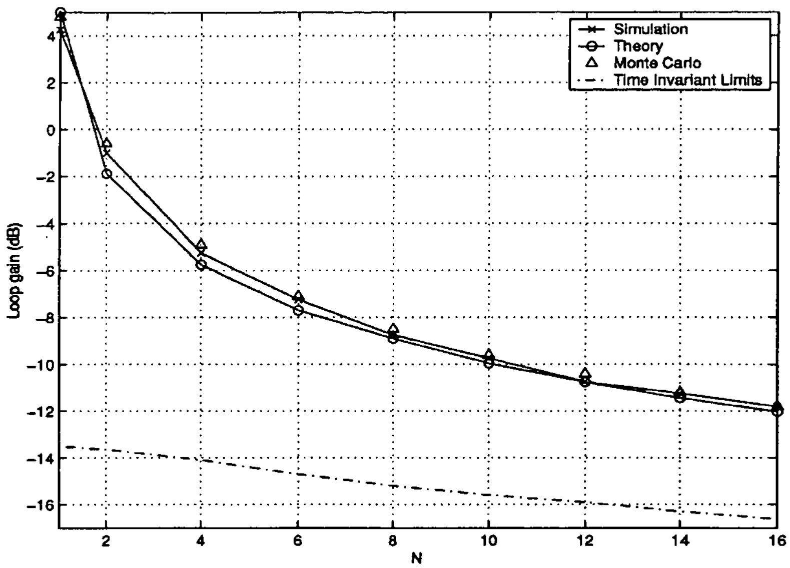

FIG. 12. Stability limits for multichannel sound systems including nonunitary reverberators, including Monte Carlo results and theoretical time-invariant loop gain limits for 50% risk of instability.

beration levels with distance mean that the loop gain in each channel may not be the same.[42] In the time-invariant case, this increases the variance of the characteristic functions,[2] and leads to a reduction in stability margin. Simulations were therefore undertaken in which the loop gains varied uniformly from $-\sigma_{\mathrm{dB}}$ to σ_{dB} dB about the chosen value. The loop gains were then normalized so that the linear mean equalled the chosen value. Figure 14 shows the output decays for a sixteen channel system with -3 to 3 dB variation in loop gain for each case. The stability limit is close to -12.05 dB, which is only 0.1 dB higher than the -11.95 dB limit in Fig. 9. This shows that frequency-shifting allows the stability limit with moderate loop gain variations to approach that for the mean loop gain across channels. However, the stability is likely to be more significantly affected for larger loop gain variations. For example, if one loop gain is set to zero the system has $N-1$ channels and Fig. 10 shows that in this case the mean loop gain limit will increase.

Finally, simulations were also undertaken in which the secondary reverberator was unitary, for $N=16$ and $N=8$ channels. A set of output envelopes for $N=16$ are shown in Fig. 15. The limit is closely similar to the decays in Fig. 9, with a stability limit of -12 dB. This is to be expected since for $N=16$ both unitary and nonunitary reverberators produce the same stability limit. The variation in the decay slopes with loop gain is slightly reduced compared to the no-reverberator case but is not as low as the standard reverberator case, however the variation in each unitary envelope is smaller than those of the no-reverberator case. This may be due to the fact that the unitary reverberator is diffusing the feedback signals from the primary reverberator, which reduces the effect of dominant feedback paths that occur with no reverberator. The envelope fluctuations are also smaller than those of the nonunitary reverberator which is probably a

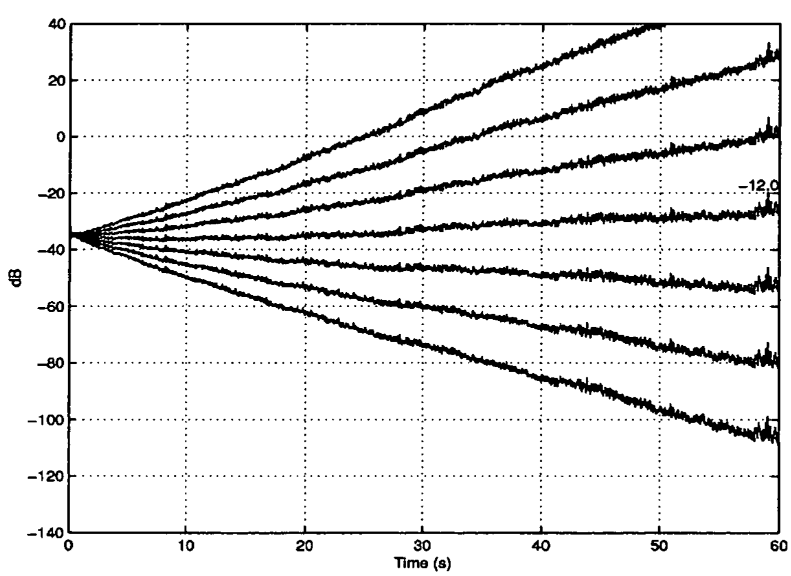

FIG. 13. Set of output envelopes for a sixteen channel system without reverberator, with loop gains -12.15 dB to -11.85 dB in 0.05 dB steps and with different frequency shifts in each channel.

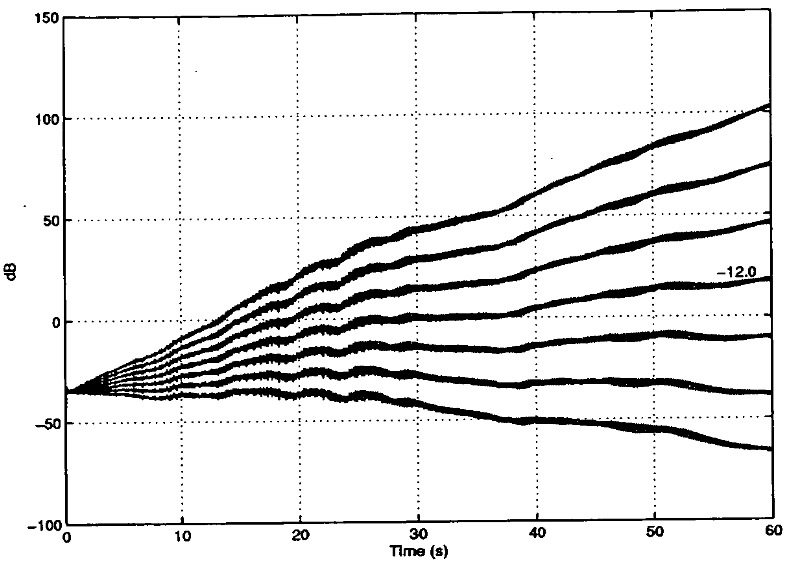

FIG. 14. Set of output envelopes for a sixteen channel system without reverberator, with loop gains -12.15 dB to -11.85 dB in 0.05 dB steps and with loop gain variation -3 to $+3$ dB across channels for each case.

result of the reduced variance in the unitary case.

The $N=8$ channel results were also similar to the no-reverberator case with an estimated stability limit of between -8.8 and -9 dB in both cases, close to the -9 dB theoretical limit.

D. Stability of time-invariant multichannel systems with reverberators

In order to assess the effectiveness of frequency-shifting, the stability limits must be compared to those obtained for multichannel time-invariant systems.[23] The stability limits for unitary systems have been derived in Ref. 2, and included in Fig. 10. However, the stability results for multichannel sound systems with a reverberator in the feedback loop—which require knowledge of the statistics of the eigenvalues of the product of two complex normal matrices—are unknown. The risk of instability (and the 50% limits used in Fig. 12) are determined empirically here using Monte Carlo simulations.

The loop transfer function matrix in systems including reverberators is the product of the room transfer function matrix and the reverberator matrix. Therefore, the stability of these systems may be estimated by generating a separate multichannel autoregressive process representing each matrix. The eigenvalues of the matrix product at each frequency are determined, and if the real part of any eigenvalue exceeds one, the system is assumed to be unstable.

For the single channel case, the calculation of eigenvalues is not required and a theoretical stability limit can be calculated. In Ref. 2 the stability of a single channel system with secondary reverberator was determined using a magnitude analysis. In Appendix D the real part analysis is carried

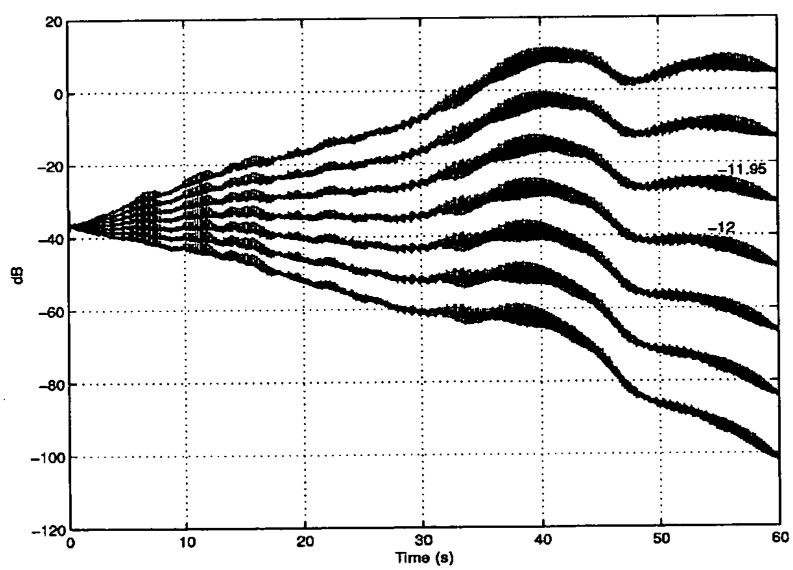

FIG. 15. Set of output envelopes for a sixteen channel system with a unitary reverberator and with loop gains from -12.15 dB to -11.85 dB in 0.05 dB steps.

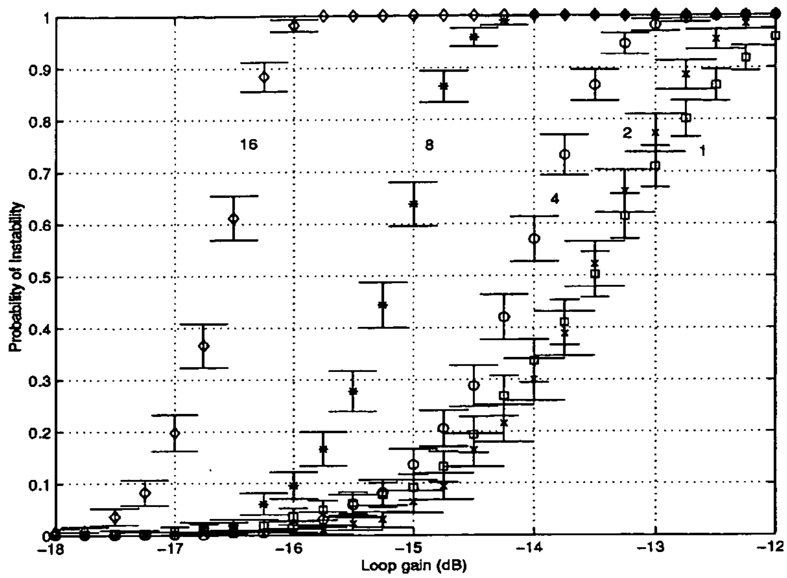

FIG. 16. Monte Carlo simulations of the probability of instability for time-invariant systems with nonunitary reverberator feedback: $N=1(\square)$, $N=2(\times)$, $N=4(\bigcirc)$, $N=8(*)$, and $N=16(\diamond)$.

out, which is more accurate. The resulting theoretical limit of -13.6 dB was closely matched by the Monte Carlo simulation result of -13.5 dB.

The probability of instability for channels in powers of two up to 16 is shown in Fig. 16, and—for comparison—the Monte Carlo results for the unitary case are shown in Fig. 17.

Note first that the unitary 50% stability limits have a 6 dB range from -9 dB ($N=1$) to -15.2 dB ($N=16$), whereas the nonunitary results have a reduced range of 3 dB from -13.5 dB to -16.6 dB. The two results become increasingly similar for increasing N, but at low channel numbers, unitary systems show an increasing 50% stability limit over the nonunitary case of up to 4.5 dB for $N=1$. (This 4.5 dB difference using the real part analysis is more accurate than the magnitude-based analysis in Ref. 2, which predicted a 5 dB difference between unitary and nonunitary cases.) The reduced performance of nonunitary systems is due to the larger variance in the eigenvalues of these systems, particularly at low channel numbers.

Secondly, the nonunitary results have a reduced slope with loop gain, showing a higher risk of instability for a given number of dB below the 50% limit than the unitary case. In other words, nonunitary time-invariant systems are likely to require a larger stability margin than unitary systems. This is again due to the increased variance of the eigenvalues, and is similar to the FS results in Figs. 2 and 3.

Thirdly, the stability limits for the non-unitary case are approximately the same for $N=1$ and $N=2$ channels. This is due to the fact that the real part of the eigenvalues of the $N=2$ system have similar statistics to the real part of the $N=1$ case.

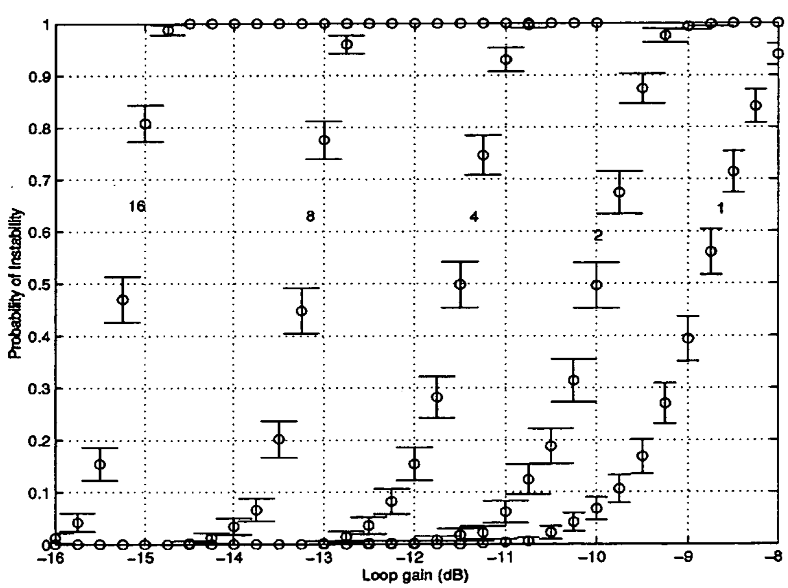

FIG. 17. Monte Carlo simulations of the probability of instability for time-invariant systems with unitary reverberator feedback.

TABLE I. Time-invariant, unitary feedback.

N	50% Limit	Impulse	Music	Minimum	Margin
1	−9.0	−15 (2.5)	−14 (5)	−15	6
2	−10.2	−15 (2)	−15 (3)	−15	5
4	−11.6	−17 (1)	−17 (4)	−17	6
8	−13.3	−19 (2.5)	−20 (3)	−20	7
16	−15.2	−21 (1.5)	−21 (3)	−21	6

IV. PRELIMINARY ASSESSMENT OF SUBJECTIVE LOOP GAIN LIMITS

Sound systems must always be operated at some margin below the absolute stability limit to sound acceptable. The theoretical limits derived above do not provide sufficient information to determine these loop gain limits. Subjective tests were therefore carried out to estimate them. Both impulse response and music sound sources were used, to allow the assessment of both running music and transient decays. For the frequency-shifting simulations a 5 Hz shift was used. The same simulator was used for the time-invariant case with a frequency shift of 0 Hz.

The reverberation times in both reverberators were kept constant with frequency. This simplification allowed the subjective results to be compared directly with the stability limits, which also assume constant reverberation time with frequency. Three researchers made assessments of the acceptable loop gains with impulse responses, and four researchers assessed the music simulations. The results were averaged and rounded to the nearest decibel. The number of subjects is relatively small, and the results are therefore indicative only. A more rigorous subjective analysis would be required to produce more accurate results, but such tests are beyond the scope of this paper.

Stereo impulse responses were generated from adjacent outputs from the reverberator for $N=1$ to 16 channels in powers of 2. For each N, five sets of simulations were carried out for the time-invariant case, with different randomly generated matrices in the reverberators to produce a range of ringing conditions. One of these impulse responses was used to filter the music excerpt to produce a set of music samples. For the time-varying case three simulations were done, with the assumption that the results would be more homogeneous due to the averaging of the transfer functions. For each simulation, the impulse response was generated for a range of loop gains from near the stability limit to several dB below.

For the music simulations, a 10 s excerpt from Handel's water music was used as the source.[40] This includes a bass note of close to 82 Hz (E2) which allows the low frequency performance to be heard.

TABLE II. Time-invariant, nonunitary feedback.

N	50% Limit	Impulse	Music	Minimum	Margin
1	−13.5	−21 (2)	−18 (6)	−21	8
2	−13.6	−21 (2)	−20 (7)	−21	8
4	−14.1	−21 (2)	−20 (5)	−21	7
8	−15.2	−23 (1)	−22 (4)	−23	8
16	−16.6	−23 (2)	−23 (3)	−23	6

TABLE III. Frequency-shifting, unitary feedback.

N	50% Limit	Impulse	Music	Minimum	Margin
1	2.5	−8 (2)	−12 (8)	−12	15
2	−2.4	−10 (1)	−14 (4)	−14	12
4	−5.9	−11 (1)	−15 (1)	−15	9
8	−9.0	−13 (1)	−18 (2)	−18	9
16	−12.0	−16 (2)	−21 (1)	−21	9

The time-invariant stereo impulse responses had cross-correlation coefficients less than 0.5 and so were reasonably binaurally dis-similar. The stereo wave files were evaluated using headphones.

A. Results

The subjective loop gain limits determined from the impulse responses and music are shown in Tables I–IV and the estimated increases in loop gain due to frequency-shifting are shown in Table V. Also given in Tables I–IV are the range of the subjective loop gain estimates in parentheses. The ranges are typically greater for the music samples, showing that artefacts are more difficult to discern with a continuous and complex music signal.

The subjective loop gain limit for the time-invariant unitary $N=1$ case with music is -14 dB, whereas the limit with impulses was -15 dB. These results are, respectively, 2 and 3 dB lower than the -12 dB often quoted for single channel systems.[1] One reason for this is that these results are from simulations with flat loop gains and reverberation times with frequency, whereas the results in Ref. 1 are based on listening tests with actual sound systems, where the loop gain tends to reduce with frequency due to loudspeaker power responses and air absorption. In a simulation with damped reverberation times in both reverberators, but with the same low frequency loop gain, the ringing was subjectively reduced. Hence loop gains closer to -12 dB are more likely to be achieved in practical installations.

For the time-invariant unitary case the minimum subjective loop gain limit for $N=1$ is -15 dB and with frequency-shifting it is -12 dB. This 3 dB improvement is smaller than Schroeder's value of 6 dB.[23] However, the loop gain increase for the impulse case is 7 dB which is closer to Schroeder's value. This suggests that music signals offer a more stringent test for the quality of sound with frequency-shifting and that smaller loop gain improvements are possible with music than for signals with fewer continuous tones or less low frequency content, such as impulsive or speech signals.

For increasing numbers of channels, the available increase in loop gain tends to be reduced for both unitary and nonunitary cases, aside from the anomalous reduction to 1

TABLE IV. Frequency-shifting, nonunitary feedback.

N	50% Limit	Impulse	Music	Minimum	Margin
1	5	−14 (4)	−13 (8)	−14	19
2	−1.9	−15 (3)	−14 (5)	−15	13
4	−5.8	−15 (2)	−15 (2)	−15	9
8	−8.9	−17 (1)	−18 (1)	−18	9
16	−12.0	−18 (2)	−21 (1)	−21	9

TABLE V. Loop gain improvement due to frequency-shifting.

N	Unitary loop gain increase	Nonunitary loop gain increase
1	3	7
2	1	6
4	2	6
8	2	5
16	0	2

dB for the $N=2$ unitary case, which is attributed to the range of subjective estimates for the music samples and the relatively small number of subjects used.

B. Discussion

The subjective limits determined here show that—while frequency-shifting provides significant increases in theoretical loop gain—the gain increases that can be achieved without introducing artefacts are smaller. This is consistent with the results in Ref. 23. Furthermore, in keeping with the theoretical stability limits, the subjective potential gain due to frequency-shifting reduces with the number of channels. For example, for unitary systems with 16 channels, the time-invariant loop gain and frequency-shifting limits are both about -21 dB and no benefit is produced by FS. For the nonunitary 16 channel case the benefit is only 2 dB. This lack of loop gain improvement is due to the pitch-shifting of low frequencies that occurs in music which limits the loop gain to well below the theoretical value. If the frequency shift was reduced at low frequencies, the loop gain could be increased, producing some possible benefit.

Low frequency pitch-shifting artefacts may be reduced by using other time-variation methods such as delay-modulation, which produces lower pitch shifts at low frequencies. However, this also means that the low frequencies are not shifted sufficiently to produce the stability limits that frequency-shifting produces. In essence, at low frequencies, the requirement for low pitch-shifting means that the system becomes quasi-time-invariant, leading to low frequency instability, as has been reported by Nielsen and Svensson.[29] Such systems could be viewed as a low frequency time-invariant part and a high frequency time-varying part. The stability limits then become a complicated function of the two contributing parts.

Other time-variation systems may introduce both amplitude and phase modulation. Amplitude modulation has the effect of modulating the loop gain at each frequency, whereas phase modulation shifts each frequency by (in general) a varying amount with time. The subjective effect of such time variation will undoubtedly differ from that produced by frequency-shifting.

Therefore, while the theoretical maximum loop gains derived here provide an upper bound for any time-varying system, the subjective stability limits determined apply specifically to frequency-shifting, and it is expected that different subjective limits would be obtained for other forms of time-variation.

V. CONCLUSIONS

This paper has determined the theoretical stability limits of multichannel sound systems which employ frequency-shifting and compared the results with the time-invariant case. Frequency-shifting produces the best-case stability for time-varying systems since it produces infinite carrier suppression and one-sided sidebands which avoids shifting frequencies back to the carrier. It therefore provides an upper limit to the loop gain increase that may be expected from the application of time-variation methods in sound systems.

The derived stability limits show that the improvement in loop gain produced by frequency-shifting reduces with the number of channels, N, since the power gain statistics become increasingly ideal with N. Multichannel sound systems with additional nonunitary reverberation systems have an inherently lower stability limit without time-variation, and therefore time-variation provides a greater improvement in loop gain for these systems, particularly at low channel numbers.

Preliminary estimates of the subjective loop gain limits have also been made, based on both impulse responses and a music sample. The results show a reducing benefit of frequency-shifting with channel numbers, and are consistent with the theoretical limits.

Simulations have been included for the case of different frequency shifts and loop gains in each channel. These show that the theoretical stability limit is robust to such variations, although wide variations of frequency shift or loop gain can be expected to produce a degradation in performance. The subjective assessment of these variations has not been investigated.

The theoretical stability limits have been derived assuming ideal room transfer functions with zero-mean complex normal statistics (Rayleigh magnitudes), for the case of both unitary and nonunitary (Rayleigh) feedback. The analysis could be extended to include the case where one of the room transfer function matrices has nonzero means (Ricean magnitudes), producing a squared matrix norm which is noncentral χ-squared distributed.[41] This case has not been considered here, but the normalized Ricean case will produce a stability limit between the unitary and Rayleigh cases,[2] and at high channel numbers, the two stability limits become identical.

Practical sound systems have frequency-dependent loop gains due to factors such as air absorption and loudspeaker power responses. Since frequency-shifting produces a stability limit governed by the mean loop gain, it is possible that time-variation will produce greater benefits in practical systems due to the increased loop gain variance. However, the peak loop gains will also be higher, which may produce increased subjective artefacts at frequencies near the peaks. Furthermore, if the loop gain has wide variations with frequency, equalization would be more appropriate than the use of time-variance.

Frequency-shifting—while it produces the maximum possible loop gain increase—may not be subjectively optimum, since it produces significant pitch-shifting artefacts at low frequencies which limit the loop gain increase. Other time-variance techniques, while producing lower theoretical

stability limits, may be able to operate with smaller loop gain margins and sound more natural, although it remains unclear to what extent low frequency stability may be controlled with any time-varying system. However, since the maximum loop gain increase due to time-variance reduces with the number of channels, the potential increase for any time-varying system is limited for systems with large numbers of channels.

APPENDIX A: NORM EQUALITY

Consider an $N \times N$ matrix $\mathbf{H}$ with entries which are complex random variables which are zero-mean, normally distributed with identical variances σ_R^2 in the real and imaginary parts. The total variance for each complex r.v. is $\sigma^2 = 2\sigma_R^2$. The expected value of the Frobenius norm of $\mathbf{H}$ is

$$E\{\|\mathbf{H}\|^2\} = E\left\{ \sum_{n=1}^{N} \sum_{m=1}^{N} |H_{nm}|^2 \right\}$$
$$= \sum_{n=1}^{N} \sum_{m=1}^{N} E\{|H_{nm}|^2\} = N^2\sigma^2. \tag{A1}$$

Consider now a product of M independent, identically distributed matrices

$$\mathbf{H}_M = \prod_{m=1}^{M} \mathbf{H}_m . \tag{A2}$$

The elements of $\mathbf{H}_M$ each consist of a sum of N^{M-1} terms, and each term is a product of M terms. The variance of a product of M zero mean terms has a variance which is the product of the individual variances, or σ^{2M}. The sum of N^{M-1} of these terms has a variance of $N^{M-1}\sigma^{2M}$. Therefore the norm of $\mathbf{H}_M$ is

$$E\{\|\mathbf{H}_M\|^2\} = N^{M+1}\sigma^{2M}. \tag{A3}$$

The norm of a single matrix raised to the power of M is

$$E\{\|\mathbf{H}\|^2\}^M = N^{2M}\sigma^{2M}. \tag{A4}$$

Hence

$$E\{\|\mathbf{H}_M\|^2\} = \frac{1}{N^{M-1}} E\{\|\mathbf{H}\|^2\}^M. \tag{A5}$$

The norm approximation is obtained by removing the expectation

$$\|\mathbf{H}_M\|^2 \approx \frac{1}{N^{M-1}} \prod_{m=1}^{M} \|\mathbf{H}_m\|^2 \tag{A6}$$

since taking the expected values of both sides produces Eq. (A5).

APPENDIX B: RELATIONSHIP BETWEEN THE MEAN OF THE LOG AND LOG OF THE MEAN OF χ-SQUARED VARIABLES

Let $\mathbf{H}$ be an $N \times N$ matrix with elements which are zero-mean, normally distributed, with real and imaginary part variances $\sigma_I^2 = \sigma_R^2$. The squared Frobenius norm of $\mathbf{H}$ is

$$y = \|\mathbf{H}\|^2 = \sum_{n=1}^{N} \sum_{m=1}^{N} (H_{nmR}^2 + H_{nmI}^2), \tag{B1}$$

where H_{nmR} is the real part of H_{nm} and H_{nmI} the imaginary part. The norm thus consists of a sum of $2N^2$ normal random variables squared. The probability density of the sum is χ squared with $2N^2$ degrees of freedom[41]

$$p(y) = \frac{1}{2^{N^2}\sigma_R^{2N^2}\Gamma(N^2)} y^{N^2-1} e^{-y/2\sigma_R^2}. \tag{B2}$$

The mean of y is

$$E\{y\} = 2\sigma_R^2 N^2. \tag{B3}$$

The variance of y is

$$\mathrm{var}\{y\} = (2\sigma_R^2)^2 N^2. \tag{B4}$$

Consider now the mean and variance of $z = \ln(y)$. The mean of z can be calculated using the formula[34]

$$E\{g(y)\} = \int_0^\infty g(y)p(y)dy, \tag{B5}$$

where $g(y) = \ln(y)$. Substituting for $g(y)$ and $p(y)$,

$$E\{\ln(y)\} = \frac{1}{2^{N^2}\sigma_R^{2N^2}\Gamma(N^2)} \int_0^\infty \ln(y) y^{N^2-1} e^{-(y/2\sigma_R^2)} dy. \tag{B6}$$

Using the integral from Ref. 37 [4.352]

$$\int_0^\infty y^n e^{-\mu y} \ln(y) dy = \frac{n!}{\mu^{n+1}} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - C - \ln(\mu) \right], \tag{B7}$$

where $C = 0.577215$ is Euler's constant, this may be simplified to

$$E\{\ln(y)\} = \left[\sum_{k=1}^{N^2-1} \frac{1}{k} - C \right] + \ln(2\sigma_R^2)$$
$$= \Psi(N^2) + \ln(2\sigma_R^2), \tag{B8}$$

where Ψ (.) is the psi function (Appendix C). For unit power gain transfer functions $2\sigma_R^2 = 1$ and $E\{\ln(y)\} = \Psi(N^2)$.

The variance of $z = \ln(y)$ requires the expected value of z^2, which may be found by substituting $(\ln(y))^2$ in Eq. (B6) Using Ref. 37 [4.352]

$$\int_0^\infty x^{\nu-1} e^{-\mu x} (\ln(x))^2 dx = \frac{\Gamma(\nu)}{\mu^\nu} \{[\Psi(\nu) - \ln(\mu)]^2 + \zeta(2,\nu)\}, \tag{B9}$$

where ζ is the zeta function (Appendix C), yields

$$E\{(\ln(y))^2\} = \zeta(2,N^2) + (\Psi(N^2))^2. \tag{B10}$$

Hence the variance for unit power gain is

$$\mathrm{var}\{\ln(y)\} = \zeta(2,N^2). \tag{B11a}$$

The difference between $\ln(E(y))$ and $E(\ln(y))$ is

$$\ln(E(y)) - E(\ln(y)) = 2\ln(N) - \Psi(N^2)$$

$$= 2\ln(N) - \left[K - \sum_{k=1}^{N^2-1} \frac{1}{k^2} \right]. \quad \text{(B11b)}$$

In dB, $10\log(E\{y\}) = 20\log(N)$ for unit power gain, and so the difference between $\overline{10\log[\|\mathbf{H}(\omega)\|^2]}$ and $10\log[\overline{\|\mathbf{H}(\omega)\|^2}]$ is

$$\Delta(N) = 20\log(N) - \frac{10}{\ln(10)}\Psi(N^2)$$

$$= 20\log(N) - \frac{10}{\ln(10)}\left[\sum_{k=1}^{N^2-1} \frac{1}{k} - C \right]. \quad \text{(B12)}$$

APPENDIX C: PSI AND ZETA FUNCTIONS

The psi function has the series definition[37]

$$\Psi(x) = -C - \sum_{k=0}^{\infty} \left[\frac{1}{x+k} - \frac{1}{k+1} \right], \quad \text{(C1)}$$

where $C = 0.577215$ is Euler's constant. For integers $x = n$ this becomes

$$\Psi(n) = -C + \sum_{k=1}^{n-1} \frac{1}{k}. \quad \text{(C2)}$$

The Riemann Zeta function has the series definition[37]

$$\zeta(z,q) = \sum_{k=0}^{\infty} \frac{1}{(q+k)^z}. \quad \text{(C3)}$$

For integers $z = n$ and $q = m$ this can be written

$$\zeta(n,m) = \sum_{k=1}^{\infty} \frac{1}{k^n} - \sum_{k=1}^{m-1} \frac{1}{k^n}. \quad \text{(C4)}$$

For $n = 2$ the infinite sum is approximately

$$K = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1.644924 \quad \text{(C5)}$$

and

$$\zeta(2,m) = K - \sum_{k=1}^{m-1} \frac{1}{k^2}. \quad \text{(C6)}$$

APPENDIX D: STABILITY OF A SINGLE CHANNEL SOUND SYSTEM WITH REVERBERATOR

Consider a single channel sound system with a room transfer function between the power amplifier input and microphone preamplifier output $H(\omega)$ and a reverberator with transfer function $X(\omega)$. We assume that both of these have real and imaginary parts with are zero-mean normal with the same variances $\sigma_R^2 = 0.5$, so that they have unit power gain. The preamplifier output is then scaled by μ to control stability. The loop gain transfer function (excluding μ) is HX. For stability, the Nyquist theorem requires that the locus of HX does not encircle the point $1 + j0$. For a statistical analysis we assume this is equivalent to saying that the real part of HX does not exceed 1.

The real part of the loop transfer function is

$$\mathrm{Re}\{HX\} = H_R X_R - H_I X_I, \quad \text{(D1)}$$

where the subscripts R and I denote real part and imaginary part, respectively. This term is the sum of two products of normals. It may be shown that the pdf of a product of normal random variables $z = xy$ each with the same variance σ_R^2 has the form[41]

$$p(z) = \frac{1}{\pi\sigma_R^2} K_0\left(\frac{|z|}{\sigma_R^2} \right) \quad \text{(D2)}$$

and the characteristic function is

$$\phi_1(\omega) = \frac{1}{\sqrt{1 + (\omega\,\sigma_R)^2}}. \quad \text{(D3)}$$

The sum of two products of normal random variables $v = x_1 y_2 + x_2 y_2$ therefore has the characteristic function

$$\phi_2(\omega) = \frac{1}{1 + (\omega\,\sigma_R)^2} \quad \text{(D4)}$$

and the resulting pdf is

$$p(v) = \frac{1}{2\sigma_R^2} e^{-|v|/\sigma_R^2}. \quad \text{(D5)}$$

The cumulative distribution is

$$C(v) = \begin{cases} \frac{1}{2} e^{v/\sigma_R^2}, & v < 0 \\ 1 - \frac{1}{2} e^{-v/\sigma_R^2}, & v \geq 0 \end{cases}. \quad \text{(D6)}$$

Assuming that the transfer function is sampled with spacing Δf sufficient to produce uncorrelated samples, across a bandwidth B, the total number of uncorrelated samples is $B/\Delta f$, and the risk of instability with loop gain μ is[2]

$$\Pr\{\text{inst}\} = 1 - \left[C\left(\frac{1}{\mu} \right) \right]^{B/\Delta f} = 1 - \left[1 - \frac{1}{2} e^{-2/\mu} \right]^{B/\Delta f}. \quad \text{(D7)}$$

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